

Gaussian coordinate systems for the Kerr metric

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We present the whole class of Gaussian coordinate systems for the Kerr metric. This is achieved through the uses of the relationship between Gaussian observers and the relativistic Hamilton-Jacobi equation. We analyze the completeness of this coordinate system. In the appendix we present the equivalent JEK formulation of General Relativity – the so-called quasi-Maxwellian equations – which acquires a simpler form in the Gaussian coordinate system. We show how this set of equations can be used to obtain the internal metric of the Schwarzschild solution, as a simple example. We suggest that this path can be followed to the search of the internal Kerr metric.

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I. INTRODUCTION

The recognition that natural processes cannot be influenced by any choice of a representation of the events occurring in spacetime conducted the covariance principle to be assumed as one of the fundamentals of modern physics. This idea was explicitly used to build a theory of gravity by Einstein, General Relativity, which gave a step forward by invoking that the MMG (*Manifold Mapping Group*) should be taken as an invariance principle of the theory. In practical uses however, one is obliged to select a particular language by choosing a special coordinate system to describe a given phenomenon. Among all the possible choices one can make – most of them dictated by symmetry of the given problem – there is a very special one that can bring us more physical insight about the problem to be treated, that is the *Gaussian coordinate system* or *Synchronic coordinate system*. In fact, this coordinate system was suggested by C. Gauss in his works about curves and surfaces. In Gaussian coordinate system (GCS), a foliation of spacetime is made in such a way that one separates space and time, as in pre-relativistic theories. The time-like world-line of an observer that is orthogonal to the 3-d space (which is identified to the co-moving system) is such that the proper time of such observer coincides with the coordinate time.

In the standard procedure made by the father founders of relativistic cosmology, the Gaussian coordinate system appears closely related to the Cosmological Principle: the large scale structure of the Universe behaves as being homogeneous and isotropic. The solutions of Einstein equations which satisfies this postulate possess a complete Gaussian coordinate system, in addition the Gaussian surface is a Cauchy surface.

On the other hand, there are cosmological solutions which do not satisfy this postulate. However these solu-

tions have undesirable properties, as for instance, closed timelike curves.

A similar approach was done for Gödel's metric by one of us [1]. It was shown that the Gaussian coordinate system is limited and cannot be extended beyond a certain region, the domain of which depends only on the vorticity present in this geometry. Such inaccessible region defines a frontier for time-like geodesics, which prohibits the extension of the GCS into the whole manifold.

Mathematically, a Gaussian coordinate system is constructed by the definition of a hypersurface $S = S(x^\mu)$, which satisfies

$$\begin{aligned} g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} &= 1, \\ g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial \bar{x}^i}{\partial x^\nu} &= 0, \end{aligned} \quad (1)$$

where \bar{x}^i are coordinates lying on S . The Gaussian coordinates are given by $\bar{x}^\mu = (S, \bar{x}^i)$. Eq. (1) imposes $\bar{g}^{00} = 1$ and $\bar{g}^{0i} = 0$.

The main purpose of this work is to exhibit a Gaussian coordinate system for the Kerr metric. The method we use is provided by the relativistic Hamilton-Jacobi formalism of canonical transformations. The key idea is to identify the principal Hamilton function with the proper time of a test particle in this geometry. A immediate application of this method can be done for other solutions of Einstein equation as Schwarzschild and Kerr-Newman. Finally, in appendix A we list the coordinate systems encountered in literature for Kerr metric including our Gaussian coordinate system and in appendix B we regain from the *Quasi-Maxwellian formalism* (JEK equations together with the evolution equations of the kinematical quantities) the Schwarzschild solution in a Gaussian coordinate system. This formalism is an equivalent way to obtain the results of General Relativity.

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II. CANONICAL TRANSFORMATIONS FORMALISM

The action for a free test particle with mass m in the presence of a gravitational field can be written as

$$S = -m \int ds. \quad (2)$$

Assuming that only one extreme of the path is fixed and the particle is moving on a geodesic line, we make the variational principle and discover how S depends on the coordinates x^μ . Such functional form is

$$\delta S = -m u_\alpha \delta x^\alpha, \quad (3)$$

where u_α is the 4-velocity. Defining the 4-momentum

$$p_\alpha \doteq -\frac{\partial S}{\partial x^\alpha}, \quad (4)$$

it satisfies

$$p_\alpha p^\alpha = m^2. \quad (5)$$

Replacing Eq. (4) into (5), we find the relativistic Hamilton-Jacobi equation for a test particle in a gravitational field given by

$$g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} - m^2 = 0. \quad (6)$$

III. EQUIVALENCE BETWEEN A GAUSSIAN SYSTEM AND THE HAMILTON-JACOBI EQUATION

By a canonical transformation of coordinates of a physical system, so that (q, p) goes to (Q, P) via a generating function $F_1(q, Q, t)$, we have the following constraint

$$dF_1 = \sum_i (p_i dq_i - P_i dQ_i) + (K - H) dt. \quad (7)$$

In the covariant form, we write it as

$$dF_1(q^\mu, Q^\mu) = -p_\alpha dq^\alpha + P_\mu dQ^\mu, \quad (8)$$

where $p_\alpha = (H, -\vec{p})$ and $P_\mu = (K \partial t / \partial Q^0, -\vec{P})$. Using another generating function $F_2(q^\mu, P_\nu)$, such that

$$F_2 = F_1 - P_\mu Q^\mu, \quad (9)$$

we have

$$dF_2 = -p_\alpha dq^\alpha - Q^\mu dP_\mu. \quad (10)$$

Then the function F_2 must satisfy

$$\begin{aligned} p_\alpha &= -\frac{\partial F_2}{\partial q^\alpha}, \\ Q^\mu &= -\frac{\partial F_2}{\partial P_\mu}. \end{aligned} \quad (11)$$

Comparing with Eq. (4), we identify $F_2(q, P)$ as the action $S(q, P) = S(q, p(q, P))$ of a test particle in a gravitational field. Therefore,

$$p_\alpha = -\frac{\partial S}{\partial q^\alpha}, \quad (12a)$$

$$Q^\mu = -\frac{\partial S}{\partial P_\mu}. \quad (12b)$$

Assuming $Q^0 = -S/m$, from the 0-component of Eq. (12b), we get

$$\frac{S}{m} = \frac{\partial S}{\partial P_0}, \quad (13)$$

and then,

$$S = e^{P_0/m} f(q^\mu, P_i). \quad (14)$$

The Hamilton-Jacobi equation can be obtained making the new Hamiltonian K constant ($H \rightarrow K \equiv \text{const}$) and, as we know that $P_0 \propto K$, it is possible to incorporate this constant into S and it follows that $S = S(q^\mu, P_i)$. Hereupon, rewriting Eq. (6) we obtain

$$g^{\mu\nu} \frac{\partial Q^0}{\partial x^\mu} \frac{\partial Q^0}{\partial x^\nu} = 1. \quad (15)$$

We derive partially Eq. (15) with respect to P_i and use Eq. (12b) to get

$$g^{\mu\nu} \frac{\partial Q^0}{\partial x^\mu} \frac{\partial Q^i}{\partial x^\nu} = 0. \quad (16)$$

Note that the coordinate system Q^μ together with Eqs. (15) and (16) define a Gaussian coordinate system.

IV. APPLICATIONS

A. Gaussian system for the Schwarzschild metric

As a simple exercise, we exhibit a Gaussian coordinate system found from the definition (1) for internal and external Schwarzschild solution. This case is particularly simple due to the symmetries of such metric.

1. The external case

Considering Schwarzschild external geometry described in the usual coordinate system, for radial observers with non null velocity at the infinity [2], the metric which is originally given by

$$ds^2 = \left(1 - \frac{r_H}{r}\right) dt^2 - \left(1 - \frac{r_H}{r}\right)^{-1} dr^2 - r^2 d\Omega^2, \quad (17)$$

where $r_H = 2M$ and M is the geometrical mass of the gravitational source, becomes

$$ds^2 = d\tau^2 - \left(\alpha^2 - 1 + \frac{r_H}{r}\right) dR^2 - r(\tau, R)^2 d\Omega^2, \quad (18)$$

according to the following coordinate transformation

$$\begin{cases} \tau = \alpha t + F_e(r, \alpha), \\ R \doteq \frac{\partial \tau}{\partial \alpha}, \end{cases} \quad (19)$$

where $\alpha \in \mathfrak{R}$ is a external parameter and τ is interpreted as the proper time. Substituting this proposal in Eq. (1) one can see that $F'_e(r, \alpha) \equiv dF_e/dr$ must satisfy

$$F'_e(r, \alpha) = \sqrt{\frac{\alpha^2 - (1 - \frac{r_H}{r})}{(1 - \frac{r_H}{r})^2}}, \quad (20)$$

for the new coordinate system to be admissible.

Analyzing the particular case in which the velocity is zero [3], we obtain the coordinate transformation integrating the following geodesic equations parameterized by the proper time τ

$$\begin{cases} \frac{d^2 t}{d\tau^2} + \frac{A'}{A} \frac{dt}{d\tau} \frac{dr}{d\tau} = 0, \\ \frac{d^2 r}{d\tau^2} + \frac{1}{2} A' A \left(\frac{dt}{d\tau}\right)^2 - \frac{1}{2} \frac{A'}{A} \left(\frac{dr}{d\tau}\right)^2 = 0, \\ \frac{d\theta}{d\tau} = 0, \\ \frac{d\phi}{d\tau} = 0, \end{cases} \quad (21)$$

where $A = (1 - r_H/r)$ and we obtain

$$\begin{cases} t = \tau + r_H \left[\ln \left(\frac{\sqrt{r/r_H} + 1}{|\sqrt{r/r_H} - 1|} \right) - 2\sqrt{\frac{r}{r_H}} \right], \\ r = \left[-\frac{3}{2}\sqrt{r_H}(\tau + R) \right]^{2/3}. \end{cases} \quad (22)$$

The new metric is obtained just choosing $\alpha^2 = 1$ in Eq. (18). It means that we are describing particles whose mechanical energy is equal to the rest energy. From this we conclude that $\alpha^2 \geq 1$. Note that this coordinate transformation does not converge to the inverse coordinate

transformation of (19) if we choose $\alpha^2 \rightarrow 1$ before the calculations, because we lose the degree of freedom necessary to build the other coordinates. In both cases we observe that the horizon “disappears”, i.e., this coordinate system does not have any problem for $r = r_H$ as in Eddington-Finkelstein or Kruskal-Szekeres coordinates system. However, in these cases, the natural “observers” are null-type. On the other hand, in the Gaussian systems (19) and (22) the true natural observers $V^\mu = \delta_0^\mu$ are time-like in every point, making possible the description of events of the spacetime from geodesic massive particles at rest.

2. The internal case

To construct a Gaussian system for the interior solution, we can consider a spherical shell filled with a perfect fluid of energy density $\rho \equiv \text{const.}$ and pressure $p = 0$ co-moving to $u_\mu = (e^{\nu/2}, 0, 0, 0)$. In this case the line element is given by

$$ds^2 = e^{\nu(r)} dt^2 - \left(1 - \frac{r_H}{r}\right)^{-1} dr^2 - r^2 d\Omega^2, \quad (23)$$

where

$$e^{\nu(r)} = \left(\frac{3}{2} \sqrt{1 - \frac{r_0^2}{r_c^2}} - \frac{1}{2} \sqrt{1 - \frac{r^2}{r_c^2}} \right)^2, \quad (24)$$

and r_0 is the radius of the “star” and $r_c = 3/\rho$. We also assume that $r_0 < r_c$ to avoid singularities at the coordinate system. Similar to the coordinate transformation (19), we have

$$\begin{cases} \tau = \alpha t + F_i(r, \alpha), \\ R = \frac{\partial \tau}{\partial \alpha}, \end{cases} \quad (25)$$

where $F'_i(r, \alpha) \equiv dF_i/dr$ must satisfy

$$F'_i(r, \alpha) = \sqrt{(\alpha^2 e^{-\nu} - 1) \left(1 - \frac{r_H}{r}\right)}. \quad (26)$$

Therefore, the new line element for Schwarzschild interior solution, written in Gaussian coordinates, is

$$ds^2 = d\tau^2 - (\alpha^2 - e^\nu) dR^2 - r(\tau, R)^2 d\Omega^2. \quad (27)$$

Of course, at $r = r_0$ Eqs. (17) and (23) must be the same. Then we obtain that $r_H = r_0^3/r_c^2$. The necessary continuity condition of the metric on the star shell, in Schwarzschild coordinates, is naturally carried to the Gaussian coordinates.

B. Kerr Metric

In 1963 Kerr found an exact solution [4] of Einstein equations, which describes the external space-time generated by a “source” having geometrical mass M and angular momentum a per unit of geometrical mass. In 1968, Carter developed a method to obtain all geodesics for this metric using the relativistic Hamilton-Jacobi equation [5]. In this paper, he considers the principal Hamilton function S as a function of the proper time τ of a test particle. Here we identify both.

In the appendix we exhibit (see [6]), some known coordinate systems of the Kerr metric and their main characteristics. We include in this list the Gaussian system presented here.

The line element for Kerr solution in Boyer-Lindquist coordinates is given by

$$ds^2 = \left(1 - \frac{2Mr}{\rho^2}\right) dt^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 + \frac{4Mar \sin^2 \theta}{\rho^2} dt d\phi + \left[(r^2 + a^2) \sin^2 \theta + \frac{2Mar^2 \sin^4 \theta}{\rho^2} \right] d\phi^2, \quad (28)$$

where $\rho^2 = r^2 + a^2 \cos^2 \theta$ and $\Delta = r^2 + a^2 - 2Mr$.

A simple way to calculate the geodesic equations parameterized by the proper time for Kerr solution is to use the Euler-Langrange equations found in [7]. From first integrals of these equations we construct the tangent vector field

$$\begin{aligned} V^0 &= \frac{1}{\rho^2 \Delta} [\Sigma^2 E - 2Mr a L], \\ V^1 &= \frac{F_n(r)}{\rho^2} \Delta, \\ V^2 &= \frac{G_n(\theta)}{\rho^2}, \\ V^3 &= \frac{1}{\rho^2 \Delta} [2Mr(aE - L \csc^2 \theta) + \rho^2 L \csc^2 \theta], \end{aligned} \quad (29)$$

where $\Sigma^2 = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta$. The functions $F_n(r)$ and $G_n(\theta)$ are given by

$$\begin{aligned} F_n(r) &= \pm \frac{\sqrt{(E\tilde{r}^2 - aL)^2 - \Delta(r^2 + \gamma)}}{\Delta}, \\ G_n(\theta) &= \pm \sqrt{\gamma - a^2 \cos^2 \theta - \left(Ea \sin \theta - \frac{L}{\sin \theta}\right)^2}, \end{aligned} \quad (30)$$

where $\tilde{r}^2 = r^2 + a^2$ and E , L and γ are constants of integration.

Let us suppose that the new time coordinate is represented by S , a principal Hamilton function and written as follows

$$S = -Et + L\phi + W_1(r) + W_2(\theta) \equiv -mQ^0. \quad (31)$$

Substituting Eq. (31) in Eq. (15), for $m = 1$, we obtain

$$\begin{aligned} E^2 \frac{\Sigma^2}{\rho^2 \Delta} - \frac{\Delta}{\rho^2} \left(\frac{dW_1}{dr} \right)^2 - \frac{1}{\rho^2} \left(\frac{dW_2}{d\theta} \right)^2 + \\ - 2 \frac{(2Mr - q^2)}{\rho^2 \Delta} EL - \frac{\Delta - a^2 \sin^2 \theta}{\rho^2 \Delta \sin^2 \theta} L^2 = 1, \end{aligned} \quad (32)$$

where E is interpreted as being the energy and L as being the angular momentum for a test particle when we consider the asymptotically flat regime. We can rewrite this expression in a more convenient form like that

$$\begin{aligned} -\Delta \left(\frac{dW_1}{dr} \right)^2 + \frac{(E\tilde{r}^2 - aL)^2}{\Delta} - r^2 = \\ = \left(\frac{dW_2}{d\theta} \right)^2 + \left(Ea \sin \theta - \frac{L}{\sin \theta} \right)^2 + a^2 \cos^2 \theta \equiv \gamma, \end{aligned} \quad (33)$$

where γ is a constant (of separability). So, we have two equations, one for W_1' and other for W_2' as

$$\begin{aligned} \frac{W_1}{dr} &= \pm \frac{\sqrt{(E\tilde{r}^2 - aL)^2 - \Delta(r^2 + \gamma)}}{\Delta}, \\ \frac{W_2}{d\theta} &= \pm \sqrt{\gamma - a^2 \cos^2 \theta - \left(Ea \sin \theta - \frac{L}{\sin \theta}\right)^2}. \end{aligned} \quad (34)$$

Note that $dW_1/dr = F_n(r)$ and $dW_2/d\theta = G_n(\theta)$.

The other coordinates, which we call here $Q^1 \doteq R$, $Q^2 \doteq \Theta$ and $Q^3 \doteq \Phi$, are calculated from the spatial components of the second equation of (12) by

$$\begin{cases} R \doteq -\frac{\partial S}{\partial E} = t - \frac{\partial W_1}{\partial E} - \frac{\partial W_2}{\partial E}, \\ \Theta \doteq -\frac{\partial S}{\partial L} = -\phi - \frac{\partial W_1}{\partial L} - \frac{\partial W_2}{\partial L}, \\ \Phi \doteq -\frac{\partial S}{\partial \gamma} = -\frac{\partial W_1}{\partial \gamma} - \frac{\partial W_2}{\partial \gamma}. \end{cases} \quad (35)$$

The remaining metric components can be given in the form

$$\begin{aligned} \bar{g}_{11} &= 1 - \frac{2Mr}{\rho^2} - E^2, \\ \bar{g}_{12} &= -EL - \frac{2Mar \sin^2 \theta}{\rho^2}, \\ \bar{g}_{13} &= -2[E(\gamma - a^2) + aL], \\ \bar{g}_{22} &= -(L^2 + \tilde{r}^2 \sin^2 \theta) - \frac{2Mar^2 \sin^4 \theta}{\rho^2}, \\ \bar{g}_{23} &= -2[L(r^2 + \gamma) - (E\tilde{r}^2 - aL)a \sin^2 \theta], \\ \bar{g}_{33} &= -4(r^2 + \gamma)(\gamma - a^2 \cos^2 \theta). \end{aligned} \quad (36)$$

The determinant of this new metric $\bar{g}_{\mu\nu}$ is

$$\bar{g} \doteq \det \bar{g}_{\mu\nu} = -4f(r)h(\theta), \quad (37)$$

where $f(r) = (E\tilde{r}^2 - aL)^2 - \Delta(r^2 + \gamma)$ and $h(\theta) = (\gamma - a^2 \cos^2 \theta) \sin^2 \theta - (Ea \sin^2 \theta - L)^2$. If we guarantee that $f, h > 0$ in some region of Kerr spacetime, then the metric $\bar{g}_{\mu\nu}$ will be well-defined for these events.

The usual coordinate systems for Kerr metric present problems in some regions, which are called *horizons*. In Boyer-Lindquist coordinates, for example, the mathematical expression for the Kerr horizons are

$$r_{\pm} = M \pm \sqrt{M^2 - a^2} \quad (38)$$

where $r_{+(-)}$ is called outer (inner) horizon. Differently, our Gaussian system presents a complete regularity at the metric components in the horizons. Note that there exists a divergence at the real singularity $r = 0$ and $\theta = \pi/2$.

We choose a $\bar{V}^\mu \doteq \delta_0^\mu$ in this coordinate system, which is obviously geodetic. Taking the inverse coordinate transformation, we write the observers field in Kerr coordinates in such way that

$$\begin{aligned} V^0 &= \frac{1}{\rho^2 \Delta} [\Sigma^2 E - 2Mr a L], \\ V^1 &= \frac{W'_1}{\rho^2} \Delta, \\ V^2 &= \frac{W'_2}{\rho^2}, \\ V^3 &= \frac{1}{\rho^2 \Delta} [2Mr(aE - L \csc^2 \theta) + \rho^2 L \csc^2 \theta]. \end{aligned} \quad (39)$$

If we compare Eq. (39) with Eq. (29) obtained from the Euler-Langrange equations we conclude that they are the same. Therefore, δ_0^μ corresponds to all tangent vectors of timelike geodesics for Kerr solution.

V. COMPLETENESS OF THE GAUSSIAN COORDINATE SYSTEM

At present there is not a theorem that specifies the necessary and sufficient conditions of existence of a complete Gaussian coordinate system for an arbitrary metric or whether the hyper-surface defined by it is a Cauchy surface [9].

In our case, it is obvious the impossibility to identify S with a Cauchy surface due to the presence of closed time-like curves. What about the completeness of such Gaussian system? To answer this question one can analyze eventual divergences in the expansion factor ϑ . This will allow us to recognize the regions where a given congruence of observers remains well-defined or not. A given congruence of observers is nothing but a given choice of the parameters E , L and γ . The explicit expression of the expansion is

$$\begin{aligned} \vartheta &= \frac{\cos \theta}{\rho^2 \sqrt{h}} [\gamma - a^2 (\cos^2 \theta + (2E^2 - 1) \sin^2 \theta) + 2EaL] + \\ &\quad - \frac{1}{\rho^2 \sqrt{f}} [2Er(E\tilde{r}^2 - aL) - r\Delta - (r^2 + \gamma)(r - M)], \end{aligned} \quad (40)$$

where f and h were defined at Sec. [IV B]. So at the singularity $r = 0$ and $\theta = \pi/2$ we obtain

$$\vartheta_s = \lim_{\theta \rightarrow \pi/2} \frac{[\mathcal{Q} - a^2(E^2 - 1) + L^2]}{a^2 \cos \theta \sqrt{\mathcal{Q}}} - \lim_{r \rightarrow 0} \frac{\gamma M}{ar^2 \sqrt{-\mathcal{Q}}}, \quad (41)$$

where $\mathcal{Q} \doteq \gamma - (Ea - L)^2$. We conclude that the divergence $\vartheta \rightarrow -\infty$ at the singularity is guaranteed just for the congruence which have $\mathcal{Q} = 0$. All the other congruences cannot reach the singularity.

If we pick out a given black hole with $M = 2$ and $a^2 = 1$ (fixed that $M^2 > a^2$, these numbers are completely arbitrary and they do not interfere at the results), we can make a more detailed analysis controlling the boundaries of the functions $f(r)$ and $h(\theta)$. Substituting $x = \sin^2 \theta$ in $h(\theta)$, we get

$$h(x) = (1 - E^2)x^2 + (\gamma - 1 + 2EL)x - L^2. \quad (42)$$

By the roots of this polynomial function, we can encounter intervals of the parameters range such that we can cover all values of θ . Thus, we can choose $L = 0$ and $E^2 > 1$, and consequently we obtain $E^2 \leq \gamma$. By the other hand the $f(r)$ function becomes

$$f(r) = (E^2 - 1)r^4 + 4r^3 + (2E^2 - 1 - \gamma)r^2 + 4\gamma r + (E^2 - \gamma). \quad (43)$$

From the zero-order term in r , we see that $E^2 \geq \gamma$ is a necessary condition for $f(r)$ to be greater than 0 for all values of r . Therefore, we conclude that is impossible to cover all manifold events with only one congruence of observers. The case $\mathcal{Q} = 0$ provides the only congruence which can cover all values of θ and all positive values of r .

VI. GENERALIZATION (KERR-NEWMAN METRIC)

In 1965 E. Newman et al. [8] found a generalization of Kerr solution, which describes a black hole with geometrical mass M , angular momentum a per unit of geometrical mass and charge q .

The line element for Kerr-Newman solution is

$$\begin{aligned} ds^2 &= \left(1 - \frac{(2Mr - q^2)}{\rho^2}\right) dt^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 + \\ &\quad + \frac{2(2Mr - q^2)a \sin^2 \theta}{\rho^2} dt d\phi + \\ &\quad - \left[(r^2 + a^2) \sin^2 \theta + \frac{(2Mr - q^2)a^2 \sin^4 \theta}{\rho^2}\right] d\phi^2, \end{aligned} \quad (44)$$

where $\rho^2 = r^2 + a^2 \cos^2 \theta$ and $\Delta = r^2 + a^2 + q^2 - 2Mr$.

If we make all steps used in the previous section, at the end we find the covariant metric components for Kerr-Newman solution described by a Gaussian coordinate system, which can be given as follows

$$\begin{aligned}
\bar{g}_{11} &= 1 - \frac{2Mr - q^2}{\rho^2} - E^2, \\
\bar{g}_{12} &= -EL - \frac{(2Mr - q^2)a \sin^2 \theta}{\rho^2}, \\
\bar{g}_{13} &= -2[E(\gamma - a^2) + aL], \\
\bar{g}_{22} &= -(L^2 + \tilde{r}^2 \sin^2 \theta) - \frac{(2Mr - q^2)a^2 \sin^4 \theta}{\rho^2}, \\
\bar{g}_{23} &= -2[L(r^2 + \gamma) - (E\tilde{r}^2 - aL)a \sin^2 \theta], \\
\bar{g}_{33} &= -4(r^2 + \gamma)(\gamma - a^2 \cos^2 \theta).
\end{aligned} \tag{45}$$

Now $\bar{V}^\mu \doteq \delta_0^\mu$ corresponds to all tangent vectors of timelike geodesics for Kerr-Newman solution.

If we take appropriate limits we get Gaussian coordinate systems for other metrics: in the case $a = 0$ and $q \neq 0$ we obtain a Gaussian system for Reissner-Nordström solution, if $a \neq 0$ and $q = 0$ we construct the previous Gaussian system for the Kerr metric, and finally, if $a = q = 0$ we obtain the Gaussian system for Schwarzschild case.

VII. CONCLUSIONS

Following a systematic algorithm, we have built a set of Gaussian coordinate systems for the Kerr metric and generalizations. We analyzed the completeness of the Gaussian system which is intrinsically related to the conserved quantities associated to a test observer immersed in this geometry. Another important feature is the fact that in this coordinate system the K-metric is not static, i.e., a Gaussian observer measuring the geometric properties in its neighborhoods, concludes that such metric is not static according to its own proper time. In the appendix we show how it is possible to use such GCS in order to undertake the search of the corresponding internal solution. This work is under analysis.

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Appendix A: Some Special coordinate systems for the Kerr metric

As we said, in [6] we see many coordinate systems for Kerr metric encountered in literature and the coordinate

transformations between them. Here there is a list of them, with some properties, and the inclusion of our Gaussian system.

- Kerr's original coordinates:

In this coordinate system (u, r, θ, ϕ) , the line element is

$$\begin{aligned}
ds^2 &= \left(1 - \frac{2Mr}{\rho^2}\right) du^2 - \rho^2 d\theta^2 - 2a \sin^2 \theta dr d\phi \\
&- 2dudr - \frac{4Mra \sin^2 \theta}{\rho^2} dud\phi + \\
&- \left[(r^2 + a^2) \sin^2 \theta + \frac{2Mra^2 \sin^4 \theta}{\rho^2} \right] d\phi^2,
\end{aligned} \tag{A1}$$

where $\rho^2 \equiv r^2 + a^2 \cos^2 \theta$.

The most important features in this coordinate system are:

- The appearance of a actual singularity in g_{uu} , for $r = 0$ and $\theta = \pi/2$;
- Setting $a \rightarrow 0$, the line element reduces to Schwarzschild geometry in “advanced Eddington-Finkelstein coordinates”;
- In terms of M , the line element can be put into Kerr-Schild form by $g^{\mu\nu} = g_0^{\mu\nu} + f(M, a, x^\alpha) l^\mu l^\nu$, where $g_0^{\mu\nu}$ is Minkowski spacetime and l^μ a geodetic null vector.

- Kerr-Schild “Cartesian” coordinates:

By use of (x^0, x, y, z) coordinates, the ds^2 for Kerr metric becomes

$$\begin{aligned}
ds^2 &= (dx^0)^2 - dx^2 - dy^2 - dz^2 + \\
&- \frac{2Mr^3}{r^4 + a^2 z^2} \left[dx^0 + \frac{r}{a^2 + r^2} (x dx + y dy) + \right. \\
&\left. + \frac{a}{a^2 + r^2} (y dx - x dy) + \frac{z}{r} dz \right]^2.
\end{aligned} \tag{A2}$$

The main characteristics of this coordinate system are:

- For $M \rightarrow 0$, it is Minkowski spacetime in Cartesian coordinates;
- For $a \rightarrow 0$, it is Schwarzschild solution in Cartesian coordinates;
- The full metric $(M, a \neq 0)$ is obviously the Kerr-Schild form again.

- Boyer-Lindquist coordinates:

The most useful coordinate system for Kerr metric is Boyer-Lindquist coordinates, as follows

$$\begin{aligned}
ds^2 = & \left(1 - \frac{2Mr}{\rho^2}\right) dt^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 \\
& - \left(r^2 + a^2 + \frac{2Mar}{\rho^2} \sin^2 \theta\right) \sin^2 \theta d\hat{\phi}^2 + \\
& + \frac{4Mar \sin^2 \theta}{\rho^2} dt d\hat{\phi},
\end{aligned} \tag{A3}$$

where $\Delta \equiv r^2 + a^2 - 2Mr$.

The noticeable properties are that:

- It minimizes the number of off-diagonal components of the metric;
- The asymptotic behaviour of this coordinates permits to conclude that M is indeed the mass and $J = Ma$ is the angular momentum;
- For $a \rightarrow 0$, it is Schwarzschild solution in standard coordinate system;
- For $M \rightarrow 0$, it is Minkowski line element in oblate spheroidal coordinates;
- It is a maximal extension of the Kerr manifold.

- Rational polynomial coordinates:

If we make a coordinate transformation $\chi = \cos \theta$ from Boyer-Lindquist coordinates, we have the following new version for Kerr spacetime

$$\begin{aligned}
ds^2 = & \left(1 - \frac{2Mr}{r^2 + a^2 \chi^2}\right) dt^2 - \frac{r^2 + a^2 \chi^2}{\Delta} dr^2 + \\
& - \frac{(r^2 + a^2 \chi^2)}{1 - \chi^2} d\chi^2 + \frac{4Mar(1 - \chi^2)}{r^2 + a^2 \chi^2} dt d\hat{\phi} + \\
& - (1 - \chi^2) \left(r^2 + a^2 + \frac{2Mar(1 - \chi^2)}{r^2 + a^2 \chi^2}\right) d\hat{\phi}^2.
\end{aligned} \tag{A4}$$

These coordinates introduce the following qualities:

- All metric components are rational polinomial of the coordinates;
- The non-appearance of trigonometric functions do the computational calculations faster

- Doran coordinates:

Introduced by C. Doran in 2000, here we obtain another coordinate system for Kerr metric given by

$$\begin{aligned}
ds^2 = & dt^2 - \rho^2 d\theta^2 - (r^2 + a^2) \sin^2 \theta d\phi^2 + \\
& - \frac{\rho^2}{r^2 + a^2} \left[dr + \frac{\sqrt{2Mr(r^2 + a^2)}}{\rho^2} (dt - a \sin^2 \theta d\phi) \right]^2.
\end{aligned} \tag{A5}$$

The useful features of Doran coordinates are:

- For $a \rightarrow 0$, it is Schwarzschild geometry in Painlevé-Gullstrand form;
- For $M \rightarrow 0$, we obtain Minkowski spacetime in oblate spheroidal coordinates;
- In Doran coordinates, the contravariant metric component g^{00} is equal to 1;
- According to ADM formalism, Doran coordinates slice the Kerr metric such that the “lapse” function is everywhere unity.

- Gaussian coordinates:

Constructed from the relativistic Hamilton-Jacobi equation, The line element for the Kerr Metric in a Gaussian coordinate system is

$$\begin{aligned}
ds^2 = & dT^2 - \left(E^2 - 1 + \frac{2Mr}{\rho^2}\right) dR^2 + \\
& - \left(L^2 + \tilde{r}^2 \sin^2 \theta + \frac{2Mr a^2 \sin^4 \theta}{\rho^2}\right) d\Theta^2 + \\
& - 4(r^2 + \gamma)(\gamma - a^2 \cos^2 \theta) d\Phi^2 + \\
& - \left(EL + \frac{2Mar \sin^2 \theta}{\rho^2}\right) dR d\Theta + \\
& - 2[E(\gamma - a^2) + aL] dR d\Phi + \\
& - 2[L(r^2 + \gamma) - (E\tilde{r}^2 - aL)a \sin^2 \theta] d\Theta d\Phi,
\end{aligned}$$

where $\tilde{r} = r^2 + a^2$.

Besides the usual characteristics of the Gaussian coordinates, in this case we have that

- For massive test particles, the geodesic equations parameterized by the proper time are immediately integrated;
- Differently from all other cases, the metric is non-static but it is stationary;
- This metric depends on parameters of the observers field comoving to the reference frame;

Appendix B: Schwarzschild solution from quasi-Maxwellian equations written in Gaussian coordinates

As it was showed in [10], the JEK (Jordan-Ehlers-Kundt) equations are equivalent to general relativity. Besides, they become particularly simple when expressed in a Gaussian coordinate system. Therefore, in principle, we could use this simplicity to search of an internal solution for the Kerr metric. As an example, we will apply this method to obtain the stellar Schwarzschild solution.

There are many references treating the formal deduction of these equations and their properties as for instance in [11]. These equations (JEK) can be obtained

from Bianchi's identities, together with Einstein equation, that is

$$W^{\alpha\beta\mu\nu}{}_{;\nu} = -\frac{1}{2}T^{\mu[\alpha;\beta]} + \frac{1}{6}g^{\mu[\alpha}T^{\beta]} \quad (\text{B1})$$

From this we obtain the corresponding independent projections of the divergence of Weyl tensor

$$\begin{aligned} & W^{\alpha\beta\mu\nu}{}_{;\nu} V_\beta V_\mu h_\alpha{}^\sigma, \\ & W^{\alpha\beta\mu\nu}{}_{;\nu} \eta^{\sigma\lambda}{}_{\alpha\beta} V_\mu V_\lambda, \\ & W^{\alpha\beta\mu\nu}{}_{;\nu} h_\mu{}^{(\sigma} \eta^{\tau)\lambda}{}_{\alpha\beta} V_\lambda, \\ & W^{\alpha\beta\mu\nu}{}_{;\nu} V_\beta h_{\mu(\tau} h_{\sigma)\alpha}. \end{aligned} \quad (\text{B2})$$

Besides, we also have the conservation law of the energy-momentum tensor

$$T^{\mu\nu}{}_{;\nu} = 0, \quad (\text{B3})$$

which can be projected parallel to an observer field V^μ or perpendicular to it, as it follows

$$\begin{aligned} T^{\mu\nu}{}_{;\nu} V^\mu &= 0, \\ T^{\mu\nu}{}_{;\nu} h^{\mu\alpha} &= 0, \end{aligned} \quad (\text{B4})$$

where $h^{\mu\nu} \doteq g^{\mu\nu} - V^\mu V^\nu$.

From the Riemann tensor, we can find the evolution equations for the kinematical quantities *expansion* ϑ , *shear* $\sigma_{\mu\nu}$ and *vorticity* $\omega_{\mu\nu}$ given by

$$\begin{aligned} & \dot{\vartheta} + \frac{\vartheta^2}{3} + 2(\sigma^2 + \omega^2) - a^\alpha{}_{;\alpha} = R_{\mu\nu} V^\mu V^\nu, \\ & h_\alpha{}^\mu h_\beta{}^\nu \dot{\sigma}_{\mu\nu} + \frac{1}{3} h_{\alpha\beta} (-2(\sigma^2 + \omega^2) + a^\lambda{}_{;\lambda}) + a_\alpha a_\beta + \\ & -\frac{1}{2} h_\alpha{}^\mu h_\beta{}^\nu (a_{\mu;\nu} + a_{\nu;\mu}) + \frac{2}{3} \vartheta \sigma_{\alpha\beta} + \sigma_{\alpha\mu} \sigma^\mu{}_\beta + \\ & + \omega_{\alpha\mu} \omega^\mu{}_\beta = R_{\alpha\epsilon\beta\nu} V^\epsilon V^\nu - \frac{1}{3} R_{\mu\nu} V^\mu V^\nu h_{\alpha\beta}, \\ & h_\alpha{}^\mu h_\beta{}^\nu \dot{\omega}_{\mu\nu} - \frac{1}{2} h_\alpha{}^\mu h_\beta{}^\nu (a_{\mu;\nu} - a_{\nu;\mu}) + \frac{2}{3} \vartheta \omega_{\alpha\beta} + \\ & -\sigma_{\beta\mu} \omega^\mu{}_\alpha + \sigma_{\alpha\mu} \omega^\mu{}_\beta = 0, \end{aligned} \quad (\text{B5})$$

together with the constraints

$$\begin{aligned} & \frac{2}{3} \vartheta_{;\mu} h^\mu{}_\lambda - (\sigma^\alpha{}_\gamma + \omega^\alpha{}_\gamma)_{;\alpha} h^\gamma{}_\lambda - a^\nu (\sigma_{\lambda\nu} + \omega_{\lambda\nu}) = \\ & = R_{\mu\nu} V^\mu h^\nu{}_\lambda, \\ & \omega^\alpha{}_{;\alpha} + 2\omega^\alpha a_\alpha = 0, \\ & -\frac{1}{2} h_{(\tau}{}^\epsilon h_{\lambda)}{}^\alpha \eta_\epsilon{}^{\beta\gamma\nu} V_\nu (\sigma_{\alpha\beta} + \omega_{\alpha\beta})_{;\gamma} + a_{(\tau} \omega_{\lambda)} = H_{\tau\lambda}. \end{aligned} \quad (\text{B6})$$

If we assume that Einstein equation is only valid in a Cauchy surface, the set of equations (B2)-(B6), so-called the *quasi-Maxwellian equations*, propagates it to the whole spacetime.

Let us consider a diagonal metric, similar to Schwarzschild one described in Gaussian coordinates, as follows

$$ds^2 = dT^2 - B(T, R) dR^2 - r^2(T, R) d\Omega^2, \quad (\text{B7})$$

and an observer field $V^\mu \doteq \delta_0^\mu$. The expansion ϑ for this vector is given by

$$\vartheta = \frac{1}{2} \left(\frac{\dot{B}}{B} + \frac{4\dot{r}}{r} \right), \quad (\text{B8})$$

where $\dot{Y}(T, R) \doteq \partial Y / \partial T$. After that, we calculate the shear tensor $\sigma^\mu{}_\nu$ and the electric part of Weyl tensor ($E^\mu{}_\nu \doteq -W_{\alpha\mu\beta\nu} V^\mu V^\nu$) and write them in matricial form

$$[\sigma^i{}_j] = f(T, R) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}, \quad (\text{B9})$$

$$[E^i{}_j] = g(T, R) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}, \quad (\text{B10})$$

where

$$f(T, R) = \frac{1}{3} \left(\frac{\dot{B}}{B} - \frac{2\dot{r}}{r} \right). \quad (\text{B11})$$

and

$$\begin{aligned} g(T, R) &= \frac{1}{12r^2 B^2} (-2r^2 B \ddot{B} + r^2 \dot{B}^2 - 4r B r'' + \\ & + 2r B \dot{r} \dot{B} + 2r r' B' + 4r B^2 \ddot{r} - 4B^2 - 4B^2 \dot{r}^2 + \\ & + 4B r'^2), \end{aligned} \quad (\text{B12})$$

where $Y'(T, R) \doteq \partial Y / \partial R$.

Observe that both $\sigma_{\mu\nu}$ and $E_{\mu\nu}$ are proportional. All other quantities like the magnetic part of Weyl tensor ($H_{\alpha\beta} \doteq -{}^*W_{\alpha\mu\beta\nu} V^\mu V^\nu$), vorticity $\omega_{\alpha\beta}$ and acceleration ($a^\mu \doteq V^\mu{}_{;\nu} V^\nu$) are identically zero, due to properties of the observers congruence chosen.

Let us assume that $V^\mu = \delta_0^\mu$ in Gaussian coordinates is co-moving to an arbitrary fluid, which can be expressed by

$$T_{\mu\nu} = (\rho + p) V_\mu V_\nu - p g_{\mu\nu} + q_{(\mu} V_{\nu)} + \pi_{\mu\nu} \quad (\text{B13})$$

where ρ is the energy density, p is the isotropic pressure, q_μ is the heat flux and $\pi_{\mu\nu}$ is the anisotropic pressure.

In the case of Schwarzschild stellar solution, it is assumed a perfect fluid inside a spherical shell and an accelerated observer ($u_\mu = \sqrt{g_{00}} \delta_\mu^0$) comoving to this fluid. With respect to the Gaussian observers such fluid presents a heat flux $q^\mu = (0, q^1, 0, 0)$ and an anisotropic pressure

$$[\pi^i_j] = \pi(T, R) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}. \quad (\text{B14})$$

With these considerations, the set of equations (B2) takes the form

$$\begin{aligned} g' + 3\frac{r'}{r}g &= \frac{1}{3}\rho' + \frac{\dot{r}}{r}q_1 + \frac{1}{2}\left(\pi' + 3\frac{r'}{r}\pi\right), \\ \dot{g} + 3\frac{\dot{r}}{r}g &= \frac{1}{4}f\pi - \frac{1}{2}(\rho + p)f + \frac{1}{2}\dot{\pi} + \frac{1}{6}\vartheta\pi + \\ &\quad - \frac{1}{3}\left[(q^1)' + \left(\frac{1}{2}\frac{B'}{B} - 2\frac{r'}{r}\right)q^1\right], \end{aligned} \quad (\text{B15})$$

The conservation law (B4) can be written explicitly as

$$\begin{aligned} \dot{\rho} + (\rho + p)\vartheta - \frac{3}{2}f\pi + (q^1)' + \left(\frac{1}{2}\frac{B'}{B} + 2\frac{r'}{r}\right)q^1 &= 0, \\ \pi' + 3\frac{r'}{r}\pi + q_{1,0} + \vartheta q_1 - p' &= 0. \end{aligned} \quad (\text{B16})$$

Now, the evolution of the kinematical quantities is the following

$$\begin{aligned} \dot{\vartheta} + \frac{\vartheta^2}{3} + \frac{3}{2}f^2 &= -\frac{1}{2}(\rho + 3p), \\ \dot{f} + \frac{f^2}{2} + \frac{2}{3}\vartheta f &= -g - \frac{1}{2}\pi^1_1. \end{aligned} \quad (\text{B17})$$

Finally, the only constraint equation is

$$f' + 3\frac{r'}{r}f - \frac{2}{3}\vartheta' = 0. \quad (\text{B18})$$

Hereupon the set of equations (B15)-(B18) corresponds to a problem of initial conditions, which shall give origin to Schwarzschild solution. As it is not our aim, we will not solve these equations step by step. However, we will indicate how to proceed.

First, we analyze Einstein equations for $T_{\mu\nu} = 0$ and we obviously obtain $B(T, R)$ and $r(T, R)$ like those given in Eq. (18), identifying $T = \tau$. On the other hand, from the quasi-Maxwellian equations we get

$$\begin{aligned} B &= \frac{r'^2}{1 + h(R)}, \\ \dot{r} &= \sqrt{y(R) + \frac{k}{r}}, \\ r' &= b(R)\sqrt{h(R) + \frac{k}{r}} \end{aligned} \quad (\text{B19})$$

where $h(R)$, $y(R)$ and $b(R)$ are arbitrary functions and k is a constant. If we assume as initial condition surface $r(T, R) \equiv \text{const} \rightarrow \infty$, then we will obtain the Schwarzschild external solution.

The Schwarzschild internal solution can be obtained if we consider the energy-momentum tensor associated to the Gaussian observer δ_0^μ , written in terms of the quantities associated to the observer u_μ (energy density ρ and pressure p) presented in Sec. [IV A 2], as follows

$$\begin{aligned} \rho_G &= (\rho + p)\alpha^2 e^{-\nu} - p \\ p_G &= -\frac{1}{3}[(\rho + p)(1 - \alpha^2 e^{-\nu}) - 3p] \\ q^\mu &= (\rho + p)\alpha e^{-\nu}(0, 1, 0, 0) \\ \pi &= \frac{2}{3}(1 - \alpha^2 e^{-\nu}) \end{aligned} \quad (\text{B20})$$

where $\nu = \nu(T, R)$ and α is an external parameter. Substituting these equations in the quasi-Maxwellian equations, we will find exactly the Schwarzschild internal solution with some arbitrary functions. However, we must match this solution with that coming from the initial condition, that is Einstein equation on the hyper-surface. Besides, choosing as Cauchy surface $r(T, R) = r_0$, we fix the arbitrary functions, obtaining with such a procedure the Schwarzschild stellar solution.

From this example, we conclude that the search of an internal solution for the Kerr metric most naturally should not be reduced to a perfect fluid in the Gaussian coordinate system. We should expect that the associated observer detects a heat flux, as in the case of static spherically symmetric, as above. It is clear that the quasi-Maxwellian equations for the Kerr metric in the Gaussian system is rather more involved than in the Schwarzschild case. So, as future work, we intend to modify such Gaussian coordinate system found, making some spatial coordinate transformations and then, to find an internal solution for Kerr metric using complex fluids but Gaussian observers.

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